MATC44 Week 9 Notes

1. Permutations With Repetition:

- Consider a collection of n distinct objects, $O_1, O_2, \ldots, O_n$ s.t. O_1 is repeated k_1 times, O_2 is repeated k_2 times, ..., O_n is repeated k_n times. In total, there are $(k_1+k_2+\ldots+k_n)$ objects. A permutation with repetition is an ordered rearrangement of these $(k_1+k_2+\ldots+k_n)$ objects.

- E.g. Consider the set $\{1,1,2\}$. How many different re-arrangements (permutations) of this set are there?

Soln:

There are 3 permutations with repetition. They are (1,1,2), (1,2,1) and (2,1,1). Note that we only have 3 permutations because it doesn't make sense to flip 1 and 1.

- More generally, if we have n objects s.t. k_1 of them are the same repeated element, k_2 of them are the same repeated element, ..., k_e of them are the same repeated element, then we have

$$\frac{(k_1+k_2+\ldots+k_e)!}{k_1!\cdot k_2!\cdot\ldots\cdot k_e!}$$

permutations with repetition.

2. Solutions to Linear Equations

– E.g. 1: How many solutions $(x_1, x_2, \ldots, x_n)$ to the equation $x_1 + x_2 + \ldots + x_n = k$ are there if $x_i \in \{0,1\}$?

Soln:
k of the variables must be 1 and n-k of the variables (the remaining vars) must be 0. If we choose k vars to be 1, then we automatically choose the remaining n-k vars that will be 0. Hence, it suffices to choose k of the n vars. This can be done in $\binom{n}{k}$ ways.

– E.g. 2: How many solns $(x_1, x_2, \ldots, x_n)$ to the eqn $x_1 + x_2 + \ldots + x_n = k$ are there if $x_i \in \{0, 1, \ldots, k\}$?

Soln:
We know that $k = \underbrace{1+1+\ldots+1}_{k\ 1\text{'s}}$.

This means that we need to select k vars s.t. every time we select a var, we add 1 to it. I.e. We can choose a var more than once. In fact, each var can be chosen either:
a) 0 times, in which case the value is 0, or
b) 1 time, in which case the value is 1, or
c) 2 times, in which case the value is 2, or
...
k) k times, in which case the value is k.

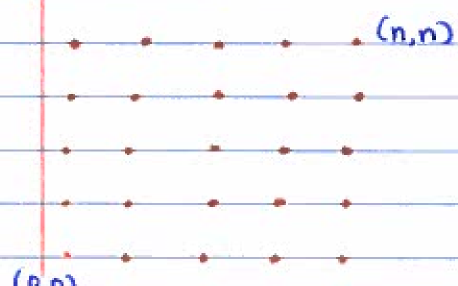
This means that we need to choose k of the n vars but we allow repetition. There are $\binom{n+k-1}{k}$ combinations with k elements with repetition, so there are $\binom{n+k-1}{k}$ different solns to the eqn.

- E.g. Suppose there's a bin with 100 red balls, 100 green balls and 100 yellow balls. How many ways can we choose 5 balls in total?

Soln:
There are $\left[\begin{matrix}3\\5\end{matrix}\right]$ or $\binom{7}{5}$ or 21 ways.

3. The Path Problem:
- Consider all points (x,y) on the plane with only int coordinates. How many diff paths are there from (0,0) to (n,n) if we can only move right, that is $(x,y) \rightarrow (x+1,y)$ and up, that is $(x,y) \rightarrow (x,y+1)$?

I.e. Consider the grid below:

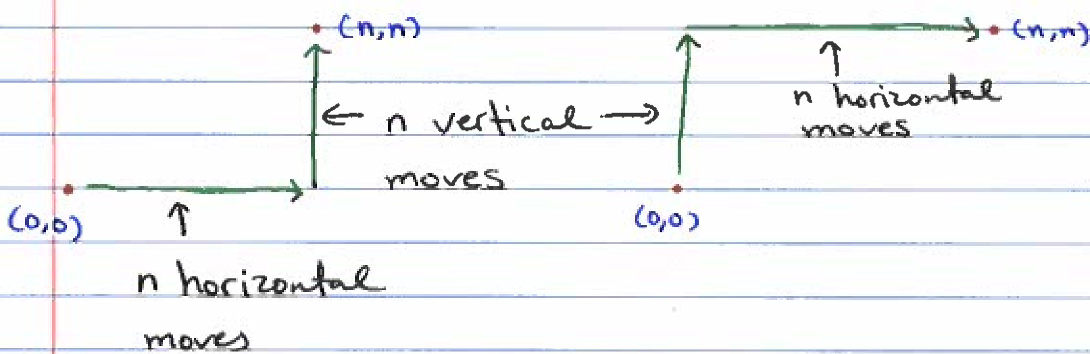


We want to find the number of paths there are from (0,0) to (n,n) if we can only move right and up.

Solution:

To get to (n,n) from $(0,0)$, we need to make n horizontal moves and n vertical moves. Therefore, there are $2n$ moves in total. Furthermore, order does not matter. If you do the n horizontal moves first followed by the n vertical moves, it's the same as if you do the n vertical moves first followed by the n horizontal moves.

I.e.



There are $2n$ moves, and you just need to choose the n horizontal moves. This is because since there are only 2 ways of moving, to the right and up, if you choose the horizontal moves, you're left with the vertical moves.

$\therefore$ The answer is $\binom{2n}{n}$.

Note: You can also get the solution using permutation with repetition.

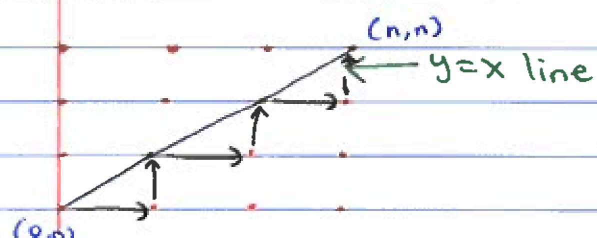
$$\frac{(n+n)!}{n!n!} = \binom{2n}{n}$$

Note: Generally, there are $\frac{(m+n)!}{m!n!}$ moves from (0,0) to (m,n), s.t. you can only move up or right.

4. The Restricted Path Problem

- Now, suppose we want to consider all paths from (0,0) to (n,n) s.t. the path cannot go over the y=x line, but can touch the y=x line.

I.e.



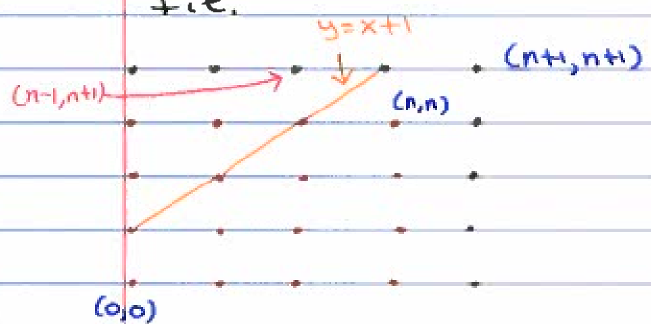
The above path is valid because it doesn't go over the y=x line.

To find number of the valid paths, we will simply subtract the number of invalid paths from the total number of paths.

We know the total number of paths is $\binom{2n}{n}$, so we just need to find the number of invalid paths.

To find the number of invalid paths, consider the following:

Let's increase the size of the grid to be $(n+1)$ by $(n+1)$. Furthermore, consider the line I.e. $y=x+1$.



We know that no valid path can touch the line $Y=X+1$. Furthermore, consider the point $(n-1,n+1)$. It is a reflection of (n,n) across the line $Y=X+1$. Note that all paths from $(0,0)$ to $(n-1,n+1)$ must cross the $y=x+1$ line. By reflecting across the line $Y=X+1$, we get a bijection between all paths from $(0,0)$ to $(n-1,n+1)$ and all invalid paths from $(0,0)$ to (n,n). Since there are $\binom{2n}{n-1}$ paths from $(0,0)$ to $(n-1,n+1)$, then there must be $\binom{2n}{n-1}$ invalid paths from $(0,0)$ to (n,n).

Therefore, the number of valid paths is equal to:

$$\binom{2n}{n} - \binom{2n}{n-1}$$

$$= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)!}$$

$$= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)(n)!}$$

$$= \frac{(2n)!}{(n)!\,n!} - \frac{(2n)!}{(n-1)!\,n!} \cdot \left(\frac{n}{n}\right) \cdot \left(\frac{1}{n+1}\right)$$

$$= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{n!\,n!} \cdot \left(\frac{n}{n+1}\right)$$

$$= \binom{2n}{n}\left[1 - \frac{n}{n+1}\right]$$

$$= \binom{2n}{n}\left(\frac{1}{n+1}\right)$$

$\therefore$ There are $\binom{2n}{n}\left(\frac{1}{n+1}\right)$ valid paths.

Note: $\binom{2n}{n}\left(\frac{1}{n+1}\right)$ is the formula that computes the Catalan Numbers.